The spectra of the oscillating shear flows.

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September 18, 2009

Abstract

We study the spectral problems for the spatially periodic flows of inviscid incompressible fluid. The basic flows under consideration are the shear flows whose profiles oscillate on high frequencies. For such flows, we present asymptotic expansions of the unstable eigenvalues in the case when the limit spectral problem has multiple eigenvalues.

Introduction

The Euler equations of inviscid incompressible fluid admit simple steady solutions known as the shear flows. Such solutions (in the Cartesian coordinates) are written in the form

$$\mathbf{V} = (U(y), 0, 0) \quad P \equiv const$$

where $\mathbf{V}$ denotes the velocity field, P denote the pressure, and U is referred to as the flow profile. Generally, function U can be chosen arbitrarily but we restrict our considerations within the case of smooth $2\pi-$periodic profiles. For such flows, we study 2D spatially periodic perturbations using the linear approximation. Let G denote the stream function for the velocity perturbation; hence the instant velocity of the perturbed flow is

$$\mathbf{V} + (G_y(x, y, t), -G_x(x, y, t), 0).$$

Then the linear equation for perturbations takes the form

$$\Delta G_t + U(y)\frac{\partial}{\partial x}\Delta G - U''\frac{\partial G}{\partial x} = 0, \ \Delta = \partial_x^2 + \partial_y^2. \tag{1}$$

Since Eq. (1) is invariant with respect to the the translations both in t and in x, it is natural to seek for

$$G(x, y, t) = e^{\sigma t} e^{i\alpha x}\Phi(y).$$

This substitution leads us to the periodic spectral problem for the Rayleigh equation

$$(\sigma + i\alpha U)\left(\frac{d^2}{dy^2} - \alpha^2\right)\Phi - i\alpha U''\Phi = 0; \quad \Phi(y) = \Phi(y + 2\pi). \tag{2}$$

If there exists an eigenvalue σ with $\mathrm{Re}\sigma > 0$ then there exists spatially periodic perturbation that grows exponentially in t and the basic shear flow is treated as unstable one. Following to [1] and [2], we look for such instability in the case of rapidly oscillating basic flow.

Let us consider profile $U(y) = W(s)$, where W – smooth 2π-periodic in s, and $s = my$ with integer m ($m >> 1$). Let us seek for the solutions of (2) in the form

$$\Phi(y) = e^{iny} f(s), \tag{3}$$

where n is integer and f is unknown 2π-periodic function. Substituting (3) into (2) yields equation for f:

$$H(\sigma, \varepsilon) f = 0, \tag{4}$$

where operator H is defined by

$$Hf = (\sigma + i\alpha W) f'' - i\alpha W'' f +$$
$$+ 2in\varepsilon(\sigma + i\alpha W) f' - \varepsilon^2(\alpha^2 + n^2)(\sigma + i\alpha W) f,$$

accent means differentiation by variable s. We construct the asymptotic for the unstable eigenvalues σ in the case of $\varepsilon = \frac{1}{m} \to 0$. Our approach follows to that of [1] and [2]. In more details, let us write the problem (4) in the form

$$H_0 f + \varepsilon H_1 f + \varepsilon^2 H_2 f = 0, \tag{5}$$
$$H_0 f = \phi f'' - \phi'' f; \quad \phi(s) = \sigma + i\alpha W(s); \tag{6}$$
$$H_1 f = 2in\phi f';; \tag{7}$$
$$H_2 f = -(\alpha^2 + n^2)\phi f;. \tag{8}$$

Assuming that $\sigma \to \sigma_0$ when $\varepsilon \to 0$ we get

$$H_{00} f \equiv (\sigma_0 + i\alpha W(s)) f'' - i\alpha W'' f = 0. \tag{9}$$

Eq. (9) has periodic solution

$$f(s) = C(\sigma_0 + i\alpha W(s))$$

for every $\sigma_0 \in \mathbb{C}$. To make this solution unique (up to the constant factor C) one have to require

$$\gamma(\sigma_0) \equiv < (\sigma_0 + i\alpha W(s))^{-2} > \neq 0, \tag{10}$$

where $<>$ denotes the averaging over the period. The analysis of [1] discovers the unstable eigenvalues under assumption (10). (In other words, the eigenvalue of the limit problem is required to be simple.) In particular, this condition is satisfied for $W(s) = \sin s$ provided $\mathrm{Re}\,\sigma_0 \neq 0$. Then σ_0 have to be selected using the solvability condition of the equation of the next approximation to the eigenfunction. For a generic profile, however, γ has zeroes outside the real axis. For such σ_0 Eq. (9) has two independent solutions, i.e. σ_0 represents a multiple eigenvalue. In this paper, we develop the asymptotic in the case of multiple σ_0. In particular, we prove that each zero of γ generically gives rise to the unique branch of *simple* eigenvalues $\sigma = \sigma(\varepsilon^2)$ while the related eigenfunction has the form $\Phi(y) = f(my)$, i.e. n must be equated to 0 in (3), (7) and (8). As an example we examine the profile $W(s) = \sin s + \sin 3s + \cos 2s$. We calculate the concrete asymptotic expansion of the unstable eigenvalues and compare it with the numeric results. The comparison exhibits very good coincidence.

It should be noted that function $\gamma(\sigma_0)$ has been introduced originally in [3] in order to formulate the necessary condition for the long-wave instability of the channel flows. However, the result of [3] exploits the monotonicity of the shear flow profile that makes it inapplicable to the oscillating flows. At the same time, the asymptotic we found can be considered as long-wave one since the problem (5) in fact depends on $\alpha^2\varepsilon^2$ only when $n = 0$.

Dispersion equation for an abstract spectral problem.

Let $H = H(\sigma, \varepsilon)$ be linear operator defined for every σ in some neighbourhood of σ_0 and for every ε in some neighbourhood of zero. Assume that domains both of H and of H^* do not depend on σ and ε, the resolvent of H is compact for every σ and ε and depends continuosly on σ and ε in uniform topology. Consider the spectral problem

$$H(\sigma, \varepsilon)f = 0, \tag{11}$$

where σ is considered as spectral parameter. Assume that $H(\sigma_0, 0) \stackrel{\text{def}}{=} H_{00}$ has non-trivial kernel: $\ker H_{00} \neq \{0\}$, i. e. σ_0 is an eigenvalue for $\varepsilon = 0$. Let us reduce the spectral problem (11) to a scalar equation for parameters σ and ε.

Operator H_{00} has compact resolvent, therefore its kernel is finite-dimensional and $\dim\ker H_{00} = \dim\ker H_{00}^*$. Let $N = \dim\ker H_{00}$, P_1, P_2 – projectors (may be non self-adjoint) onto N-dimensional subspaces $\ker H_{00}$ and $\ker H_{00}^*$. Let $Q_k = I - P_k$, $k = 1, 2$. Set

$$f = P_1 f + Q_1 f \stackrel{\text{def}}{=} \xi + \theta. \tag{12}$$

Apply P_2 and Q_2 to equation (11). Write

$$P_2 H P_1 \xi + P_2 H Q_1 \theta = 0, \tag{13}$$

$$Q_2 H P_1 \xi + Q_2 H Q_1 \theta = 0. \tag{14}$$

Operator $Q_2 H_{00} Q_1$ bijectively maps $\operatorname{Im} Q_1 \cap \mathcal{D}(H)$ on $\operatorname{Im} Q_2$. Therefore the inverse operator $(Q_2 H(\sigma, \varepsilon) Q_1)^{-1} : \operatorname{Im} Q_2 \to \operatorname{Im} Q_1$ is bounded and continuous in a small neighborhood of $(\sigma_0, 0)$. We resolve equation (14) in θ and then substitute the solution into the (13). This gives an equation in a finite-dimensional subspace $\ker H_{00}$

$$P_2 H\xi - P_2 H Q_1 (Q_2 H Q_1)^{-1} Q_2 H \xi = 0.$$

We equate the determinant of this equation to zero and arrive to a scalar equation. This is the dispersion equation for σ and ε. The structure of the asymptotic of $\sigma(\varepsilon)$ can be deduced from Newton diagram (see [4]).

The limit operator kernel

The operator $H_{00} = H(\sigma_0, 0)$ of the problem (4) is defined by

$$H_{00} f = (\sigma_0 + i\alpha W) f'' - i\alpha W'' f = \\ = \frac{d}{ds}\left((\sigma_0 + i\alpha W)^2 \frac{d}{ds} \frac{f}{\sigma + i\alpha W}\right).$$

For all σ_0 it's kernel is non-trivial. It contains functions

$$f(s) = C(\sigma_0 + i\alpha W) \tag{15}$$

with arbitrary constant C. If equation

$$\frac{d}{ds}\left((\sigma_0 + i\alpha W)^2 \frac{d}{ds}\frac{f}{\sigma + i\alpha W}\right) = 0 \tag{16}$$

doesn't have solutions different from (15) then one can apply the results [1]. We focus ourselves on the case of 2-dimensional kernel of H_{00} i.e. we assume that equation (16) has two linearly independent solutions.

Integrating equation (16), we obtain

$$\frac{d}{ds}\frac{f}{\sigma + i\alpha W} = \frac{C_1}{(\sigma + i\alpha W)^2}.$$

The solvability condition for this equation is

$$C_1 < (\sigma + i\alpha W)^{-2} >= 0,$$

where $< .. >$ denotes average: $< f >= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)\,ds$. Thus, the kernel of H_{00} is 2-dimensional if and only if

$$\gamma(\sigma_0) \equiv < (\sigma + i\alpha W)^{-2} >= 0. \tag{17}$$

Then the general solution of (16) has the form

$$\begin{aligned} \varphi_1(s) = \sigma_0 + i\alpha W(s), \quad \varphi_2(s) = \varphi_1 \int \frac{1}{\varphi_1^2} = \\ = (\sigma_0 + i\alpha W(s)) \int_0^s \frac{dy}{(\sigma_0 + i\alpha W(y))^2}. \end{aligned} \tag{18}$$

We note that $\gamma(\sigma_0)$ is analytical function provided $\operatorname{Re}\sigma_0 \neq 0$. Its zeros are the multiple eigenvalues we are looking for.

The kernel of the adjoint operator

Let σ_0 satisfy (17). Consider the adjoint operator

$$H_{00}^* f = \frac{1}{\sigma_0^* - i\alpha W}\frac{d}{ds}\left((\sigma_0^* - i\alpha W)^2 \frac{df}{ds}\right).$$

Let us integrate equation

$$\frac{d}{ds}\left((\sigma_0^* - i\alpha W)^2 \frac{df}{ds}\right) = 0,$$

and then divide the result by $(\sigma_0^* - i\alpha W)^2$, and then integrate once more. As a result we have

$$f = C_1 + C_2 \int\limits_0^s \frac{dy}{(\sigma_0^* - i\alpha W(y))^2}.$$

According to (17), f is periodic for all C_1 and C_2. Therefore, functions

$$\psi_1(s)=1,\quad \psi_2(s)=\int\frac{1}{(\varphi_1^*)^2}=\int\limits_0^s\frac{dy}{(\sigma_0^*-i\alpha W(y))^2}. \tag{19}$$

span the kernel of H_{00}^*.

Dispersion equation for the problem (4)

Let us construct dispersion equation for the case of 2-dimensional kernel of limit operator. Let P_1 and P_2 be the orthogonal projectors onto subspaces $\ker H_{00}$ and $\ker H_{00}^*$. Functions ψ_1 and ψ_2 are orthogonal so that

$$P_2f=\frac{(f,\psi_1)}{(\psi_1,\psi_1)}\psi_1+\frac{(f,\psi_2)}{(\psi_2,\psi_2)}\psi_2=<f>+\frac{(f,\psi_2)}{(\psi_2,\psi_2)}\psi_2.$$

Although φ_1 and φ_2 are not orthogonal, it is clear that projection $P_1f=0$ if and only if $(f,\varphi_1)=0$ and $(f,\varphi_2)=0$. We define projectors $Q_{1,2}$ setting $Q_{1,2}=I-P_{1,2}$.

Let $\theta=Q_1f$. Then into sum

$$\theta=\beta_1\theta^{(1)}+\beta_2\theta^{(2)}, \tag{20}$$

where functions $\theta^{(1,2)}$ satisfy an equation

$$Q_2HQ_1\theta^{(i)}+Q_2H\varphi_i=0,\quad i=1,2. \tag{21}$$

(this representation can be seen from (13)). Then the equation (14) in 2-dimensional subspace $\ker H_{00}^*$ is equivalent to

$$\begin{aligned}\beta_1\big(H(\varphi_1+\theta^{(1)}),\psi_1\big)+\beta_2\big(H(\varphi_2+\theta^{(2)}),\psi_1\big)=0,\\ \beta_1\big(H(\varphi_1+\theta^{(1)}),\psi_2\big)+\beta_2\big(H(\varphi_2+\theta^{(2)}),\psi_2\big)=0.\end{aligned}$$

The equating of the determinant of this system to zero gives us the dispersion equation

$$\Delta\equiv\begin{vmatrix}\big(H(\varphi_1+\theta^{(1)}),\psi_1\big) & \big(H(\varphi_2+\theta^{(2)}),\psi_1\big)\\ \big(H(\varphi_1+\theta^{(1)}),\psi_2\big) & \big(H(\varphi_2+\theta^{(2)}),\psi_2\big)\end{vmatrix}=0. \tag{22}$$

The construction of the asymptotic

Let's plot Newton diagram to determine the structure of the eigenvalue asymptotic (see [4]). We expand function $\theta^{(1,2)}$, operator H and determinant Δ in the powers of ε:

$$\theta^{(i)}=\theta_0^{(i)}+\varepsilon^1\theta_1^{(i)}+\varepsilon^2\theta_2^{(i)}+\ldots\quad i=1,2; \tag{23}$$

$$H(\sigma,\varepsilon)=H_0(\sigma)+\varepsilon^1H_1(\sigma)+\varepsilon^2H_2(\sigma); \tag{24}$$

$$\begin{aligned}&H_0f=\phi f''-\phi''f=\left[\phi^2\left(\frac{f}{\phi}\right)'\right]';\quad H_1f=2in\phi f';\\ &H_2f=-(\alpha^2+n^2)\phi f;\quad \phi(s)=\sigma+i\alpha W(s).\end{aligned} \tag{25}$$

$$\Delta = \Delta_0 + \varepsilon^1\Delta_1 + \varepsilon^2\Delta_2 + ... \tag{26}$$

According to expansion (24) equations (21) take the form

$$Q_2H_0\theta^{(i)}+\varepsilon^1 Q_2H_1\theta^{(i)} + \varepsilon^2 Q_2H_2\theta^{(i)} = \\ = -Q_2H_0\varphi_i - \varepsilon^1 Q_2H_1\varphi_i - \varepsilon^2 Q_2H_2\varphi_i.$$

The substitution of (23) gives series of equations for coefficients $\theta_k^{(i)}$, $i = 1, 2$

$$\begin{aligned} k{=}0: &\quad Q_2H_0\theta_0^{(i)} = -Q_2H_0\varphi_i, \\ k{=}1: &\quad Q_2H_0\theta_1^{(i)} + Q_2H_1\theta_0^{(i)} = -Q_2H_1\varphi_i, \\ k{=}2: &\quad Q_2H_0\theta_2^{(i)} + Q_2H_1\theta_1^{(i)} + Q_2H_2\theta_0^{(i)} = -Q_2H_2\varphi_i, \\ k{\geq}3: &\quad Q_2H_0\theta_k^{(i)} + Q_2H_1\theta_{k-1}^{(i)} + Q_2H_2\theta_{k-2}^{(i)} = 0. \end{aligned} \tag{27}$$

Theorem 1. *a) Coefficients $\Delta_0(\sigma)$ and $\Delta_1(\sigma)$ identically equal to zero.*
b) If $n \neq 0$ then $\Delta_2(\sigma_0) \neq 0$, and such σ_0 produce no analytical branches of eigenvalues $\sigma(\varepsilon)$.
c) If $n = 0$ then $\Delta_2(\sigma_0) = 0$ and $\Delta_{2k+1}(\sigma) \equiv 0$, $k \in \mathbb{N}$.
d) Let $n = 0$ and $\int_{-\pi}^{\pi} \varphi_1^{-3}\, ds \neq 0$. If σ_0 is real then assume additionally that $\sigma_0^2 \neq \alpha^2 < W^2 >$. Then σ_0 is a limit point for some branch of simple eigenvalues $\sigma(\varepsilon^2)$, and

$$\sigma = \sigma_0 + \sigma_2\varepsilon^2 + O(\varepsilon^4), \quad \sigma_2 = -\frac{\Delta_4(\sigma_0)}{\frac{d\Delta_2}{d\sigma}(\sigma_0)}, \tag{28}$$

where $\Delta_4(\sigma_0)$ and $\frac{d\Delta_2}{d\sigma}(\sigma_0)$ are defined by equalities

$$\Delta_4(\sigma_0) = \begin{vmatrix} (H_2\varphi_1, \psi_1) & (H_2\varphi_2, \psi_1) \\ (H_2\varphi_1, \psi_2) & (H_2\varphi_2, \psi_2) \end{vmatrix},$$

$$\frac{d\Delta_2}{d\sigma}(\sigma_0) = 2\alpha^2 < \varphi_1^2 >< \varphi_1^{-3} > .$$

To prove the theorem we have to expand the scalar products $\big(H_i\varphi_j, \psi_k\big)$ and $\big(H_i\theta_m^{(j)}, \psi_k\big)$ in the powers of ε.

Lemma 1. *For all σ: $\operatorname{Re}\sigma \neq 0$ following statements hold:*
a) $\big(H_0\varphi_j, \psi_1\big) = \big(H_0\theta_m^{(j)}, \psi_1\big) = 0$;
b) $\big(H_1\varphi_1, \psi_1\big) = 0$; c) $\big(H_1\varphi_2, \psi_1\big) = in$;
d) $\big(H_1\varphi_1, \psi_2\big) = -in$; e) $\big(H_0\varphi_1, \psi_2\big) = 0$
f) If $\sigma \neq \sigma_0$ then any solution of $Q_2H_0\xi = 0$ can be represented in the form $\xi = C_1\phi + C_2\widetilde{\xi}$, where $\widetilde{\xi}$ – particular solution of $H_0\widetilde{\xi} = \psi_2$, function ϕ defined in (25).

Proof of the lemma. Equalities a,b,c,d and e are straightforward. To get statement f one can use the equivalence of equations $Q_2H_0\xi = 0$ and $H_0\xi = \alpha_1\psi_2 + \alpha_2\psi_2$. Then $\alpha_1 = 0$ (according to the solvability condition). Then the general solution of homogeneous equation $H_0\xi = 0$ is $C_1\phi$ while $\alpha_2\widetilde{\xi}$ is particular solution of the equation $H_0\xi = \alpha_2\psi_2$. This completes the proof.

Proof of the theorem 1.

a) According to statement a of lemma 1

$$\Delta_0 = \begin{vmatrix} \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} = 0.$$

$$\Delta_1 = \begin{vmatrix} \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} +$$

$$+ \begin{vmatrix} \big(H_0\theta_1^{(1)},\psi_1\big) & \big(H_0\theta_1^{(2)},\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} +$$

$$+ \begin{vmatrix} \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} +$$

$$+ \begin{vmatrix} \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0\theta_1^{(1)},\psi_2\big) & \big(H_0\theta_1^{(2)},\psi_2\big) \end{vmatrix} =$$

$$= \begin{vmatrix} \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix}.$$

According to statement f of the lemma $\varphi_m + \theta_0^{(m)} = C_1^{(m)}\phi + C_2^{(m)}\widetilde{\xi}$. Then

$$\big(H_0(\varphi_m+\theta_0^{(m)}),\psi_2\big) = C_2^{(m)}\big(H_0\widetilde{\xi},\psi_2\big) = C_2^{(m)}\big(\psi_2,\psi_2\big),$$

$$\big(H_1(\varphi_m+\theta_0^{(m)}),\psi_1\big) = C_2^{(m)}\big(H_1\widetilde{\xi},\psi_1\big).$$

(the statement b of lemma 1 was used for the last equality). Thus, we get a determinant with proportional rows

$$\Delta_1 = \begin{vmatrix} C_2^{(1)}\big(H_1\widetilde{\xi},\psi_1\big) & C_2^{(2)}\big(H_1\widetilde{\xi},\psi_1\big) \\ C_2^{(1)}\big(\psi_2,\psi_2\big) & C_2^{(2)}\big(\psi_2,\psi_2\big) \end{vmatrix} = 0.$$

b) Writing coefficient Δ_2 similarly to Δ_1 we obtain the sum of eight determinants. Half of them are equal to zero according to the statement a of lemma 1. As a result we arrive at equality

$$\begin{aligned} \Delta_2 = &\begin{vmatrix} \big(H_2(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_2(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} + \\ &+ \begin{vmatrix} \big(H_1\theta_1^{(1)},\psi_1\big) & \big(H_1\theta_1^{(2)},\psi_1\big) \\ \big(H_0(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_0(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} + \\ &+ \begin{vmatrix} \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_2\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_2\big) \end{vmatrix} + \\ &+ \begin{vmatrix} \big(H_1(\varphi_1+\theta_0^{(1)}),\psi_1\big) & \big(H_1(\varphi_2+\theta_0^{(2)}),\psi_1\big) \\ \big(H_0\theta_1^{(1)},\psi_2\big) & \big(H_0\theta_1^{(2)},\psi_2\big) \end{vmatrix}. \end{aligned} \tag{29}$$

Let $\sigma = \sigma_0$. Then the first, second and forth determinants are equal to zero as $\operatorname{Im} H_{00} \perp \psi_{1,2}$. since $H_{00}\varphi_{1,2} = 0$ equations (27) yield $\theta_0^{(1,2)} = 0$ provided $\sigma = \sigma_0$. Using statements b,c and d of the lemma 1 we get

$$\Delta_2(\sigma_0) = \begin{vmatrix} 0 & in \\ -in & (H_1\varphi_2, \psi_2) \end{vmatrix} = -n^2.$$

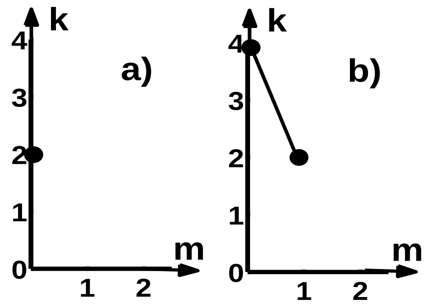

Fig. 1. Newton diagram

If $n \neq 0$ then $\Delta_2(\sigma_0) \neq 0$. The related Newton diagram is presented on fig. 1a. Line $k = 0$ corresponds to the coefficient Δ_0, line $k = 1$ — to the coefficient Δ_1 and so on. Since $\Delta_2(\sigma_0) \neq 0$, Newton diagram includes point $(0, 2)$, while there are no points below the line $k = 2$ (due to statement a of theorem 1). Therefore, equation (22) doesn't have solutions nearby σ_0 when $\varepsilon \to 0$.

c) If $n = 0$ then $H_1 = 0$ and operator H in fact depends on ε^2. Consequently, coefficients Δ_{2k+1} are equal to zero.

Let's calculate $\Delta_4(\sigma_0)$. Taking into account that $\theta_0^{(1,2)}(\sigma_0) = 0$ and $\operatorname{Im} H_{00} \perp \psi_{1,2}$, we get

$$\Delta_4(\sigma_0) = \begin{vmatrix} \big(H_2\varphi_1, \psi_1\big) & \big(H_2\varphi_2, \psi_1\big) \\ \big(H_2\varphi_1, \psi_2\big) & \big(H_2\varphi_2, \psi_2\big) \end{vmatrix}.$$

Let's calculate $\frac{d\Delta_2}{d\sigma}(\sigma_0)$. Taking into account that $H_1 = 0$ and (29) we get

$$\frac{d\Delta_2}{d\sigma}(\sigma_0) = \begin{vmatrix} \big(H_2\varphi_1, \psi_1\big) & \big(H_2\varphi_2, \psi_1\big) \\ \big(\frac{\partial H_0}{\partial\sigma}\varphi_1, \psi_2\big) & \big(\frac{\partial H_0}{\partial\sigma}\varphi_2, \psi_2\big) \end{vmatrix}.$$

Then $\big(\frac{\partial H_0}{\partial\sigma}\varphi_1, \psi_2\big) = 0$ by statement e of the lemma. Since $\frac{\partial H_0}{\partial\sigma} f = f''$, we have

$$\frac{d\Delta_2}{d\sigma}(\sigma_0) = \big(H_2\varphi_1, \psi_1\big)\big(\varphi_2'', \psi_2\big). \tag{30}$$

Let's determine scalar products $\big(H_2\varphi_1, \psi_1\big)$ and $\big(\varphi_2'', \psi_2\big)$.

$$\big(H_2\varphi_1, \psi_1\big) = -\frac{\alpha^2}{2\pi}\int_{-\pi}^{\pi} \varphi_1^2\, ds = -\alpha^2\big[\operatorname{Re}(\sigma^2) - \alpha^2 < W^2 > + i\sigma_{0_{re}}\sigma_{0_{im}}\big].$$

If $\sigma_{0_{im}} \neq 0$, then imaginary part of the scalar product is nonzero otherwise we have to require that $\sigma_0^2 \neq \alpha^2 < W^2 >$ in order ensure that $\frac{d\Delta_2}{d\sigma}(\sigma_0)$ is nonzero. Let's transform the second multiplier in formula (30)

$$\big(\varphi_2'', \psi_2\big) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi_2 \Big(\frac{1}{\varphi_1^2}\Big)' ds = \frac{1}{\pi}\int_{-\pi}^{\pi} \varphi_1 \Big(\int \frac{1}{\varphi_1^2}\Big) \frac{-\varphi_1'}{\varphi_1^3} ds =$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \Big(\frac{1}{\varphi_1}\Big)' \Big(\int \frac{1}{\varphi_1^2}\Big)\, ds = -\frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{\varphi_1^3}\, ds.$$

Assuming that $< \varphi_1^{-3} > \neq 0$ we arrive at the Newton diagram presented on fig. 1b which, in turn, implies (28). The proof is completed.

Computing experiment

Consider flow profile $W(s) = \sin s + \sin 3s + \cos 2s$. If $n = 0$, then problem (4) depends on only one parameter: after substitution $\sigma = i\alpha\widetilde{\sigma}$, $\varepsilon = \widetilde{\varepsilon}/\alpha$ only one parameter $\widetilde{\varepsilon}$ and unknown eigenvalue $\widetilde{\sigma}$ left. Therefore, we can set $\alpha = 1$ with no losses in generality. Function γ defined in (17) vanishes in the point $\sigma_0 = 0.3543 - 0.6366i$. Coefficient σ_2 of asymptotic expansion (28) is $\sigma_2 = -0.2568 + 0.2393i$. The asymptotic $\sigma_{ap} = \sigma_0 + \sigma_2\varepsilon^2 + \ldots$ demonstrates very good agreement with the Numerical results (see fig. 2 and 3).

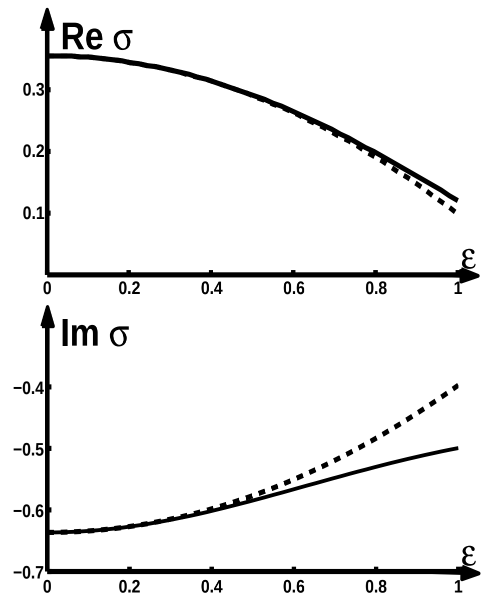

Fig. 2. Plots of the numerical (solid line) and asymptotic (dashed line) solution $\sigma(\varepsilon)$. Abscissa is ε, ordinate axises are $\mathrm{Re}\,\sigma$ and $\mathrm{Im}\,\sigma$ respectively.

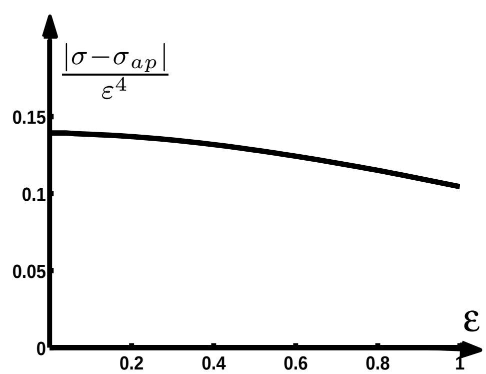

Fig. 3. The relative difference between the numerical solution and asymptotic. Abscissa is ε, ordinate axis is $\frac{|\sigma-\sigma_{ap}|}{\varepsilon^4}$.

Acknowledgements. I thank Andrey Morgulis for the problem definition, fertile conversations and great help in translation of the article. This research was performed with the support of

the U.S. Civilian Research and Development Foundation (grant RUM1-2842-RO-06) and Russian Foundation for Fundamental Research (grant 08-01-00895-a).

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